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Scanning the structure of ill-known spaces: Part 1. Founding principles about mathematical constitution of space

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Abstract. Some necessary and sufficient conditions allowing a previously unknown space to be explored through scanning operators are reexamined with respect to measure theory. Some generalized conceptions of distances and dimensionality evaluation are proposed, together with their conditions of validity and range of application to topological spaces. The existence of a Boolean lattice with fractal properties originating from nonwellfounded properties of the empty set is demonstrated. This lattice provides a substrate with both discrete and continuous properties, from which existence of physical universes can be proved, up to the function of conscious perception. Spacetime emerges as an ordered sequence of mappings of closed 3-D Ponicaré sections of a topological 4-space provided by the lattice, and the function of conscious perception is founded on the same properties. Self-evaluation of a system is possible against indecidability barriers through anticipatory mental imaging occurring in biological brain systems; then our embedding universe should be in principle accessible to knowledge. The possibility of existence of spaces with fuzzy dimension or with adjoined parts with decreasing dimensions is raised, together with possible tools for their study.

The work presented here provides the introductory foundations supporting a new theory of space whose physical predictions (suppressing the opposition of quantum and relativistic approaches) and experimental proofs are presented in details in Parts 2 and 3 of the study.

Key words: Space structure; topological distances; dimensionality assessment; Spacetime differential element; Space lattice; objects origin.

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1. Introduction

Starting from perceptive aspects, experimental sciences give rise to theoretical descriptions of hidden features of the surrounding world. On the other hand, the mathematical theory of demonstration teaches that any property of a given object, from a canonical particle to the universe, must be consistent with the characteristics of the corresponding embedding space (Bounias, 2000a). In short, since what must be true in abstract mathematical conditions should also be fulfilled upon application to the observable world, whether the concept of 'reality' has a meaning or not. Conversely, since similar predictions can infer when classical properties of Spacetime are tested against various theories, e.g. an unbounded or a bounded (non-Archimedean) algebra (Avinash and Rvachev, 2000), the most general features should be accessible without previous assumptions about peculiarities of the explored spaces.

Indeed, even the abstract branch of sciences, e.g. pure mathematics, show a tendency to going to a form of "experimental research", essentially under the pressure of some limitations that metamathematical considerations raise about the fundamental questions of decidability (Chaitin, 1998-1999). However, algorithmic information theory does not embrace the whole of mathematics, and the theory of demonstration escapes the limitations of arithmetical axiomatics, in particular through anticipatory processes (Bounias, 2000b; 2001). To which extent the real world should obey just arithmetics rules remains to be supported, and instead, the foundations of the existence and functionality of a physical world have been shown to be more widely provided by general topology (Bounias, 2000a). On the other hand, how to explore a world considered as a system has been matter of thorough investigation by Lin (1988; 1989).

The goal of the present work is to examine in depth some founding mathematical conditions for a scientific scanning of a physical world to be possible through definitely appropriate tools. In this respect, it is kept in mind that human perceptions play a part, at least since humans (which includes scientists) are self-conscious of their existence through their perception of an outside world, while they believe in the existence of this outside world because they perceive it. This "judge and party" antinomy will be addressed in the present essay.

The first part of this essay is dealing with the notions of measure and distances in a broad topological sense, including for the assessment of the dimensionality of a space whose detached pieces (i.e. the data collected through apparatus from a remote world) are scatteredly displayed on the table of a scientist, in view of a reconstruction of the original features. The existence of an abstract lattice will be deduced and shown to stand for the universe substrate (or "space"). A second part will focus on specific features of this lattice and a confrontation of this theoretical framework with the corresponding model of Krasnoholovets and Ivanovsky (1993) about quantum to cosmic scales of our observed universe will be presented, along with experimental probes of both the theory and the model. A third part will further present the structures predicted for elementary particles whose existence derives from the described model, and lead to a confrontation of the predictive performances of the various theories in course.

2. Preliminaries

2.1. About the concepts of measure and distances

Whatever the actual structure of our observable spacetime, no system of measure can be operational if it does not match with the properties of the measured objects. Scanning an light-opaque world with a light beam, by ignorance of the fundamental structure of what is explored, though a caricatural example, illustrates the principle of a necessarily failing device, those results would raise discrepancies in knowledge of the studied world. An example is given by the recent development of ultraviolet astronomy: when the sky is scanned through ultraviolet instead of visible radiation, the resulting extreme ultraviolet astrophysical picture of our surrounding universe becomes different (Malina, 2000).

One of the problems faced by modeling unknown worlds could be called *"the syndrome of polynomial adjustment"*. In effect, given an experimental curve representing the behavior of a system whose real mechanism is unknown, one can generally perform a statistical adjustment with using a polynomial system like $\mathrm{M} = \sum_{(\mathrm{i}=0 \to \mathrm{N})} \mathrm{a_i} \cdot \mathrm{x^i}$. Then, using an apparatus adjusted to test for the fitting of the N+l parameters to observational data will require increasingly accurate adjustment, so as to convincingly reflect the natural phenomenon within some boundaries, while if the real equation is mathematically incompatible with the polynomial, there will remain some irreducible parts in the fitting attempts. This might well be what occurs to the standard cosmological model and its 17 variables, with its failure below some quantum scales (Arkani-Hamed et al., 2000). Furthermore, Wu and Lin (2002) have demonstrated how the approximations of solutions of equations describing nonlinear systems mask the real structures of these systems. It may be that what is tested in accelerators is a kind of self-evaluation of the model, which poses a problem with respect to the indecidability of computed systems as successively raised by Godel (1931), Church (1934), Turing (1936), and more recently Chaitin, (1998-1999). However, mathematical limits in computed systems can be overpassed by the biological brain's system, due to its property of self-decided anticipatory mental imaging (Bounias, 2001). It will be shown here how this makes eventually possible a scanning of an unknown universe by a part of itself represented by an internal observer.

2.1.1. Measure

The concept of measure usually involves such particular features as the existence of mappings and the indexation of collections of subsets on natural integers. Classically, a measure is a comparison of the measured object with some unit taken as a standard (James and James, 1992). However, sets or spaces and functions are measurable under various conditions which are cross-connected. A mapping f of a set E into a topological space $\mathcal{T}$ is measurable if the reciprocal image of a open of $\mathcal{T}$ by f is measurable in E, while a set measure on E is a mapping $\mathfrak{m}$ of a tribe B of sets of E in the interval $[0, \infty]$, exhibiting denumerable additivity for any sequence of disjoint subsets $(\mathrm{b_n})$ of B, and denumerable finiteness, i.e. respectively:

$$\mathfrak{m}\left(\overset{\infty}{\underset{\mathrm{n=0}}{\cup}} \mathrm{b_n}\right) = \sum_{\mathrm{n=0}}^{\infty} \mathfrak{m}\,(\,\mathrm{b_n}) \tag{1.1}$$

$$\exists \text{ An}, \quad \text{An} \in \text{B}, \qquad \text{E} = \ \cup \text{An}, \quad \forall \in \mathbf{N},\ \mathfrak{m}(\text{An}) \ \text{ is finite.} \tag{1.2}$$

Now, coming to the "unit used as a standard", this is the part played by a gauge (J). Again, a gauge is a function defined on all bounded sets of the considered space, usually having non-zero real values, such that (Tricot, 1999):

(i) a singleton has measure naught: V x, J({x}) = 0;
(ii) (J) is continued with respect to the Hausdorff distance;
(iii) (J) is growing: E $\subset$ F $\Rightarrow$ J(E) $\subset$ J(F);
(iv) (J) is linear: F(r·E) : r·J(E).

This implies that the concept of distance is defined: usually, a diameter, a size, or a deviation are currently used, and it should be pointed that such distances need to be applied on totally ordered sets. Even the Caratheodory measure (μ^*) poses some conditions which again involve a common gauge to be used:

(i) A $\subset$ B $\Rightarrow$ μ(A)$\leq$ μ^*(B);
(ii) for a sequence of subsets (Ei): μ^* $\subset$(Ri) $\leq$ $\Sigma\,\mu^*$(Ri);
(iii) $\angle$(A,B), A $\cap$ B= $\emptyset$: μ^*(A $\cup$ B)= μ^*(A)+μ^*(B) (in consistency with (1.1));
(iv) μ^*(E)= μ^*(A $\cap$ E)+μ^*($\complement_{\mathrm{E}}$A $\cap$ E).

The Jordan and Lebesgue measures involve respective mappings (I) and ($\mathfrak{m}^*$) on spaces which must be provided with $\cap$, $\cup$ and $\complement$. In spaces of the $\mathbb{R}^{\mathrm{n}}$ type, tessellattion by balls are involved (Bounias and Bonaly, 1996), which again demands a distance to be available for the measure of diameters of intervals.

A set of measure naught has been defined by Borel (1912) first as a linear set (E) such that, given a number (e) as small as needed, all points of E can be contained in intervals whose sum is lower than (e).

Remark 2.1.1. Applying Borel intervals imposes that appropriate embedding spaces are available for allowing these intervals to exist. This may appear as a not explicitly formulated axiom, which might involve important consequences (see below).

2.1.2. A corollary on topological probabilities

Given a set measure (P) on a space E= ((X,A),$\perp$), (A) a tribe of parts of X, then (Chambadal, 1981): for a $\in$ A, P(a) = Probability of (a) and P(X) = 1; for A,B in X, one has P(A u B) = P(A) + P(B); for a sequence {An} of disjoint subspaces, one has: $\lim_{\mathrm{n}\to\infty}$ P(An) = 0.

A link can be noted with the Urysohn's theorem: let E, F be two disjoint parts of a metric space W: there exists a continued function f of W in the real interval [0,1] such that f(x) = 0 for any x $\in$ E, f(x) = 1 for any x $\in$ F, and 0 < f(x) > 1 in all other cases. This has been shown to define conditions providing a probabilistic form to a determined structure holding for a deterministic event (Bonaly and Bounias, 1995). In effect, if W is the embedding space and S a particular state of universe in W as recalled in section 3 below. Let C the complementary of S in W, and x an object in the set of closed sets in W. Let $I_S(x)$ be an indicative function such that $I_S(x) = 1$ if $x \in S$ and $I_S(x) = 0$ if $x \in C$. Writing $I_S(x) = P(X)$: $x \in S \Rightarrow P(x) = 1$; $x \in X \Rightarrow P(x) = 0$ and $0 < P(x) < 1$ in all other cases.

A probabilistic adjustment, as accurate as it can show, is thus not a proof that a phenomenon is probabilistic in its essence.

2.1.3. Distances

Following Borel, the length of an interval $F = [a,b]$ is:

$$L(F) = (b - a) - \sum_n L(C_n) \tag{1.3}$$

where C_n are the adjoined, i.e. the open intervals inserted in the fundamental segment.

Such a distance is required in the Hausdorff distances of sets (E) and (F): Let E(e) and F(e) the covers of E or F by balls B(x,e), respectively for $x \in E$ or $x \in F$,

$$\mathrm{dist}_H(E, F) = \inf\{e : E \subset F(e) \wedge F \subset E(e)\} \tag{1.4a}$$

$$\mathrm{dist}_H(E, F) = (x \subset E,\ y \subset F : \inf\ \mathrm{dist}(x, y)) \tag{1.4b}$$

Since such a distance, as well as most of classical ones, is not necessarily compatible with topological properties of the concerned spaces, Borel provided an alternative definition of a set with measure naught: the set (E) should be Vitali-covered by a sequence of intervals (Un) such that: (i) each point of E belongs to a infinite number of these intervals; (ii) the sum of the diameters of these intervals is finite.

However, while the intervals can be replaced by topological balls, the evaluation of their diameter still needs an appropriate general definition of a distance.

A more general approach (Weisstein, 1999b) involves a path $\varphi(x,y)$ such that $\varphi(0) = x$ and $\varphi(1) = y$.

For the case of sets A and B in a partly ordered space, the symmetric difference $\Delta(A, B) = \complement_{A\cup B}(A \cap B)$ has been proved to be a true distance also holding for more than two sets (Bounias and Bonaly, 1996; Bounias, 1997-2000). However, if $A \cup B = \emptyset$, this distance remains $\Delta = A \cup B$, regardless of the situation of A and B within an embedding space E such that $(A, B) \subset E$. A solution to this problem will be derived below in terms of a separating distance versus an intrinsic distance.

2.2. On the assessment of space dimensions

One important point is the following: in a given set of which members structure is not previously known, a major problem is the distinction between unordered N-uples and ordered N-uples. This is essential for the assessment of the actual dimension of a space.

2.2.1. Fractal to topological dimension

Given a fundamental segment (AB) and intervals Li = [Ai, A(i+1)], a generator is composed of the union of several such intervals: $G = \cup_{(i \in [1,n])}(Li)$. Let the similarity coefficients be defined for each interval by ϱi = dist(Ai, A(i+1))/dist(AB).

The similarity exponent of Bouligand is (e) such that for a generator with n parts:

$$\sum_{i \in [1,n]} (\varrho i)^{e} = 1. \tag{2.1}$$

When all intervals have (at least nearly) the same size, then the various dimension approaches according to Bouligand, Minkowsky, Hausdorff and Besicovitch are reflected in the resulting relation:

$$n \cdot (\varrho)^{e} = l, \tag{2.2a}$$

that is:

$$e \approx \operatorname{Log} n / \operatorname{Log} \varrho. \tag{2.2b}$$

When e is an integer, it reflects a topological dimension, since this means that a fundamental space E can be tessellatted with an entire number of identical balls B exhibiting a similarity with E, upon coefficient ϱ.

2.2.2. Parts, ordered N-uples and simplexes

2.2.2.1. Parts. A set is composed of members of which some are themselves containing further members. In solving the Russell and Burali-Forti paradoxes by making more accurate the definition of a set, Mirimanoff (1917) classified members in nuclei (i.e. singletons or "atoms") with no members inside themselves, and parts containing members. Then, a set E is said first kind (Ei) if it is isomorph to none of its members, and second kind (En) if it is isomorph to at least one of its members. Hence:

E = {a,b,c, (d,e,f,)} is first kind since X = (d, e, f) is not isomorph to {(a,b,c,(X)}.

F = {a, (b, (c, (Z))} is second kind since by posing H={c,(Z)} and G = {b, (H)}, it appears that F = (a,(G)} is isomorph to G = {b, (H)},as well as to H, and eventually further to members of H. Mirimanoff called a 'descent' a structure of the following form, where (E) denotes a set or part of set, and (e) a nucleus:

$$E^{(n)} = \{e^{(n)}, (E^{n+1})\}. \tag{3.1}$$

A descent is finite if none of its parts is infinitely iterated. Then, a second kind set is ordinary if its descent is finite and extraordinary if its descent include some infinite part. One can recognize a fractal feature in an extraordinary second kind set.

Remark 2.2. A finite number (n) of iterations provides a form of measure called the length of the descent. In the above example of set F, assuming Z is not isomorph to F, the sequence {F, G, H} is bijectively mapped on segment {1, 2, 3} of natural integers.

2.2.2.1. ordered N-uples. Let the members of the above set E be ordered into a structure of the type of F, for example: $F' = \{a, (b, (c, (d,e,f)))\}$.

The length of the descent of the exemplified F'' is $L(F') = 3$, and the last part is not isomorph to F'. Now, suppose that members (a, b, c, ...) are similar, that is no particular structure nor essential feature allow one to be distinguished from the other. Then, F' could be written in the alternative form: $F'' = \{\{a\},\{a,\{a,...\}\}\}$. This indicates the availability of an order to hold on F''. Usually, a part (a,b) or {a,b} is not ordered until it can be written in the form: $(ab) = \{\{a\}, \{a,b\}\}$. The nonempty part of (ab) owns a lower boundary: {a}, and an upper boundary: (ab) itself. Similarly:

$$(abc) = \{\{a\},\ \{a,(a,b)\},\ \{a,(a,b),\ (a,b,c)\}\},\ \text{etc.} \tag{3.2}$$

Stepping from a part (a,b,c,...) to a N-uple (abc) needs that singletons are available in replicates. Two cases are met: if members (a,b,c,...) are not identical, replicates can be found in the set of parts of the set. Otherwise, if members are similar singletons, then the set is just isomorphic with a segment of the set of natural integers.

In both cases, for any pair of members (i.e. as subset members or singletons), the Cartesian product will give a set of ordered pairs. Repeating the operation in turn gives ordered N-uples. A formal distinction of (xy), (x,y) and {x,y} will appear below.

2.2.2.2. Simplexes. A simplex is the smaller collection of points that allows the set to reach a maximum dimension. In a general accepting, it should be noted that the singletons of the set are called vertices and ordered N-uples are (N-1)-faces A^{N-1}. A set of (N+1) members can provide a structure of dimension at most (N), that is: a connected "(N+l)- object" has dimension $d \leq N$. The number of 1-faces A^1 will be $(N+1)N/2!$, the number of 2-faces A^2 will be $(N+1) \cdot N \cdot (N-1)/3!$, $n(A^3) = (N+1) \cdot N \cdot (N-1) \cdot (N-2)/4!$, and finally $n(A^k) = (N+1) \cdot N \cdot (N-1) ... (N-k+1)/(k+1)!$ (Banchoff, 1996).

One question emerging now with respect to the purpose of this study is the following: given a set of N points, how to evaluate the dimension of the space embedding these points?

3. Distances and dimensions revisited

3.1. The relativity of a general form of measure and distance

Our approach aims at searching for distances that would be compatible with both the involved topologies and the scanning of objects not yet known in the studied spaces. No such configuration is believed to be an exception nor a general case.

3.1.1. General case of a not necessarily ordered topological distance

Proposition 3.1. A generalized distance between spaces A,B within their common embedding space E is provided by the intersection of a path-set φ(A,B) joining each

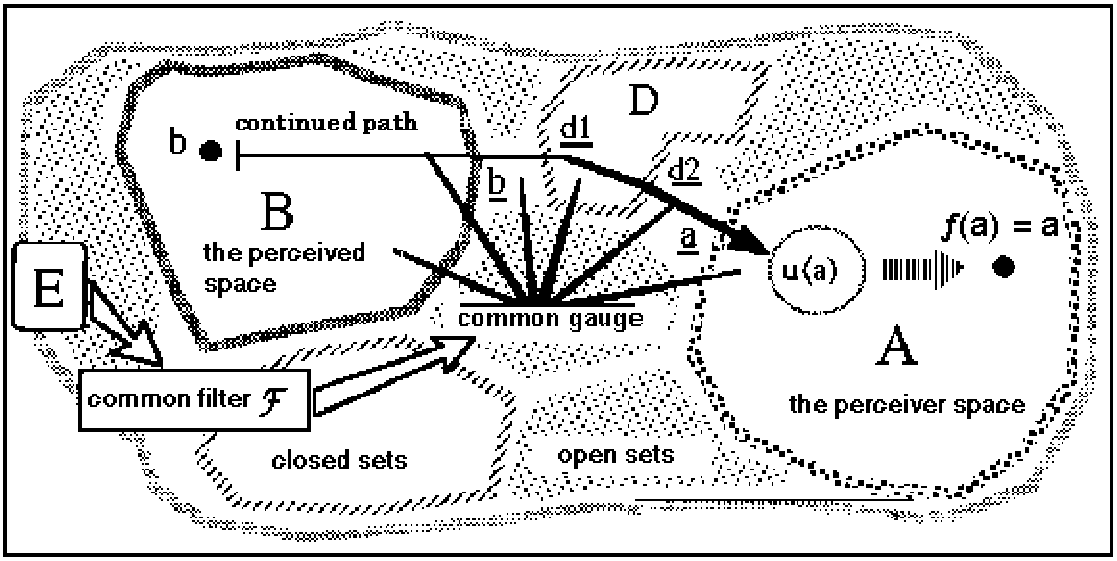

Figure 1: Some features of the measure of a distance between closed sets (A,B) in an embedding space E. Space A is the perceiver and space B is the perceived one. Points $\underline{a}$, $\underline{b}$, $\underline{d_1}$, $\underline{d_2}$ are Jordan's points.

member of A to each member of B with the complementary of A and B in E, such that: φ(A,B) is a continued sequence of a function f of a gauge (J) belonging to the ultrafilter of topologies on {E, A, B,...}.

Proof. (i) A measure comes to a reality when it maps a perceiving system (e.g. A) to a perceived one (e.g. B) so that B is said measured by A within E. It has been formerly shown that A and B should be topologically closed, since Jordan's points are needed for the characterization of a path joining any point of the interior of B to a point of the interior of A. The nonempty intersection $\underline{b}$ of the path with the frontier ∂B of B leads to the intersection $\underline{a}$ of the path with the frontier ∂A of A, and a sequence of mappings u($\underline{a}$) of ($\underline{a}$) to a fixed point f(a) = a in A provides the mathematical foundation of a mental image (Bounias and Bonaly, 1997) (Figure 1).

The path φ(A,B) is a set composed as follows: (i) $\varphi(\mathrm{A},\mathrm{B}) = \cup_{\mathrm{a}\in\mathrm{A},\,\mathrm{b}\in\mathrm{B}}\,\varphi(\mathrm{a},\mathrm{b})$, all defined in a sequence interval $[0,\ \mathrm{f}^{\mathrm{n}}(\mathrm{x})],\ \mathrm{x}\in\mathrm{E}$.

Then, for any closed D situated between A and B, $\mathrm{f}^{\mathrm{n}}(\mathrm{m})$ intersects the frontiers of B, D, and A: thus, the sequence f^{n} has some of its points identified with $\underline{b}$, ($\underline{di}$, $\underline{dj}$, ... $\in \partial$D), and $\underline{a}$. Therefore, the relative distance of A and B in E, noted $\Lambda_{\mathrm{E}}(\mathrm{A},\mathrm{B})$ is contained in φ(A,B):

$$\Lambda_{\mathrm{E}}(\mathrm{A},\mathrm{B}) \subseteq \varphi(\mathrm{A},\mathrm{B}) \tag{4.1}$$

Eventually one may have in the above case : $\min\{\Lambda_{\mathrm{E}}(\mathrm{A},\mathrm{B})\} = \{\underline{b}, \underline{di}, \underline{dj}, \ldots, \underline{a}\}$, while in all cases: $\inf\{\Lambda_{\mathrm{E}}(\mathrm{A},\mathrm{B})\} = \{\underline{b} \vee \underline{a}\}$. Denote by E° the interior of E, then:

$$\min\{\varphi(\mathrm{A},\mathrm{B}) \cap \mathrm{E}^\circ\} \text{ is a geodesic of space E connecting A to B} \tag{4.2a}$$

$$\max\{\varphi(\mathrm{A},\mathrm{B}) \cap \mathrm{E}^\circ \cup \complement_{\mathrm{E}^\circ}(\mathrm{A}\cup\mathrm{B})\} \text{ is tesselation of E out of A and B} \tag{4.2b}$$

It is noteworthy that relation (4.2a) refers to $\dim\Lambda = \dim\varphi$, while in relation (4.2b) the dimension of the probe is that of the scanned sets.

(ii) Let $\mathrm{J} = \mathrm{f\,n(m)}$, such that: $\varphi(0) = \mathrm{b}$ and $\varphi\mathrm{f^n(m)} = \mathrm{a}$. Then, $\mathrm{m} \in \mathcal{U}(\mathrm{E})$, with $\mathcal{U}(\mathrm{E})$ the ultrafilter on topologies of E.

Suppose that $\mathrm{m} \notin \mathcal{U}(\mathrm{E})$. Then, there exists a filter $\mathcal{F}_\mathrm{A}$ holding on A and a filter $\mathcal{F}_\mathrm{B}$ holding on B, such that $\mathcal{F}_\mathrm{A} \neq \mathcal{F}_\mathrm{B}$. Let $\mathrm{x} \in \mathcal{F}_\mathrm{A}$: there exists $\mathrm{y} \in \mathcal{F}_\mathrm{B}$ with $x \neq y$ and $\mathrm{y} \notin \mathcal{F}_\mathrm{A}$.

Therefore, if $\mathrm{x} \in \varphi\mathrm{f^n(m)}$, $\mathrm{y} \notin \varphi\mathrm{f^n(m)}$ and the path-set does not measure B from a perception by A. That $\mathrm{m} \in \mathcal{U}(\mathrm{E})$ is a necessary condition.

(iii) Let (O) be a open set of E. Then, a member of φ(A,B) joining B to A through O meets no frontier other than B and $\partial\mathrm{A}$, and the obtained $\Lambda_\mathrm{E}(\mathrm{A,B})$ ignores set O. As a consequence: only closed structures can be measured in a topological space by a path founded on a sequence of Jordan's points: this justifies and generalizes the Borel measure recalled above. In contrast, if a closed set D contains closed subsets, e.g. $\mathrm{D'} \subset \mathrm{D}$, then there is a member of φ(A,B) that intersects D′. If in addition the path is founded on $\mathrm{f^n(m)}$, $\mathrm{m} \in \mathcal{U}(\mathrm{E})$, there exists nonempty intersections of the type $\underline{\mathrm{d'i}}, \underline{\mathrm{d'j}} \in \mathrm{f^n(m)} \cap \partial\mathrm{D'}$. Therefore, $\Lambda_\mathrm{E}(\mathrm{A,B})$ will include $\underline{\mathrm{d'i}}$ and $\underline{\mathrm{d'j}}$ to the measured distance. This shows that $\varphi(\mathrm{A,B}) \cap \mathrm{E}^\circ$ is a growing function defined for any Jordan point, which is a characteristic of a Gauge. In addition, the operator $\Lambda_\mathrm{E}(\mathrm{B,A})$ defined by this way meets the characteristics of a Fréchet metrics, since the proximity of two points $\underline{\mathrm{a}}$ and $\underline{\mathrm{b}}$ can be mapped into the set of natural integers and even to the set of rational numbers: for that, it suffices that two members $\varphi\big(\mathrm{f^n(x)}\big), \mathrm{f^n(y)}\big)$ are identified with an ordered pair $\{\varphi\big(\mathrm{f^n(x)}\big), \{\varphi\big(\mathrm{f^n(x)}, \mathrm{f^n(y)}\big)\}\}$.

(iv) Suppose that one path φ(a,b) meets an empty space $\{\varnothing\}$. Then a discontinuity occurs and there exists some i such that $\varphi\big(\mathrm{f^i}(\underline{\mathrm{b}})\big)$. If all φ(A,B) meet $\{\varnothing\}$, then no distance is measurable. As a corollary, for any singleton $\{\mathrm{x}\}$, one has $\varphi\big(\mathrm{f}(\{\mathrm{x}\})\big) = \varnothing$. The above properties meet two other characteristics of a gauge.

Remarks 3.1. Given closed sets {A,B,C, ...}= E, then a path set $\varphi(\mathrm{E,EŒ})$ exploring the distance of E to the closure (Œ) of E meets only open subsets, so that $\varphi(\mathrm{E,EŒ}) = \varnothing$. This is consistent with a property of the Hausdorff distance. Similarly, given A,B c E, one can tentatively note:

$$\inf_{\mathrm{x}\in\mathrm{E}}\{\varphi(\mathrm{A},\{\mathrm{x}\}) \cap \mathrm{E}^\circ\} \;\mapsto\; \mathrm{dist}_{\mathrm{Hausdorff}}(\{\mathrm{x}\},\mathrm{A}) \tag{4.3a}$$

as reported by Choquet (1984) or Tricot, 1999b);

$$\sup\varphi\{(\mathrm{A,B}) \cap \mathrm{E}^\circ\} \;\mapsto\; \mathrm{dist}_{\mathrm{Hausdorff}}(\mathrm{A,B}) \tag{4.3b}$$

as reported by Tricot (1999b);

$$\inf_{\mathrm{a}\in\mathrm{A},\,\mathrm{b}\in\mathrm{B}}\varphi\{(\mathrm{a,b}) \cap \mathrm{E}^\circ\} \;\mapsto\; \mathrm{dist}(\mathrm{A} \wedge \mathrm{B}) \tag{4.3c}$$

as reported by Choquet (1984) for a (E,d) metric space;

$$\max\{(\mathrm{A,B}) \subset \mathrm{E}|\Lambda_\mathrm{E}(\mathrm{A,B})\} \;\mapsto\; \mathrm{diam}_{\mathrm{Hausdorff}}(\mathrm{E}) \tag{4.3d}$$

in all cases.

(v) $\Lambda_E(A, B) = \Lambda_E(B, A)$ and $\Lambda_E(\{x, y\}) = \emptyset \Leftrightarrow x = y$. If the triangular inequality condition is fulfilled, then $\Lambda_E(B, A)$ will meet all of the properties of a mathematical distance.

$\Lambda(A, B) \cup \Lambda_E(B, C)$ may contain members of $\Lambda_E(A, C)$ since the latter are contained in neighborhoods of A,B,C. Thus:

$$\Lambda_E(A, C) = \{\exists B, (A \cap B \neq \emptyset,\ C \cap B \neq \emptyset),\ \Lambda_E(A, C) \subseteq \Lambda_E(A, B) \cup \Lambda_E(B, C)\}. \tag{4.3e}$$

This completes the proof of Proposition 2.1.

3.1.2. The particular case of a totally ordered space

Let A and B be disjoint segments in space E. Let E be ordered by the classical relations:

$$A \subset B \ \Leftrightarrow\ A \prec B, \tag{5.1a}$$

$$(A, B) \subset \Leftrightarrow\ E \succ A,\ E \succ B. \tag{5.1b}$$

Then, E is totally ordered if any segment owns a infinum and a supremum. Therefore, a distance (d) between A and B is represented as illustrated by Figure 2 by the following relation:

$$d(A, B) \subseteq \mathrm{dist}(\inf A, \inf B) \cap \mathrm{dist}(\sup A, \sup B) \tag{5.2}$$

with the distance evaluated through either classical forms or even the set-distance Δ(A,B) which will be revisited below.

3.1.3. The case of topological spaces

Proposition 3.2. A space can be subdivided in two main classes: objects and distances.

The set-distance is the symmetric difference between sets: it has been proved that it owns all the properties of a true distance (Bounias and Bonaly, 1996) and that it can be extended to manifolds of sets (Bounias, 1997). In a topologically closed space, these distances are the open complementary of closed intersections called the "instances". Since the intersection of closed sets is closed and the intersection of sets with nonequal dimensions is always closed (Bounias and Bonaly, 1994), the instances rather stands for closed structures. Since the latter have been shown to reflect physical-like properties, they denote objects. Then, the distances as being their complementaries will constitute the alternative class: thus, a physical-like space may be globally subdivided into objects and distances as full components. This coarse classification will be further refined in Part 2.

The properties of the set-distance allows an important theorem to be now stated.

Theorem 3.1. Any topological space is metrizable as provided with the set-distance (Δ) as a natural metrics. All topological spaces are kinds of metric spaces called "delta-metric spaces".

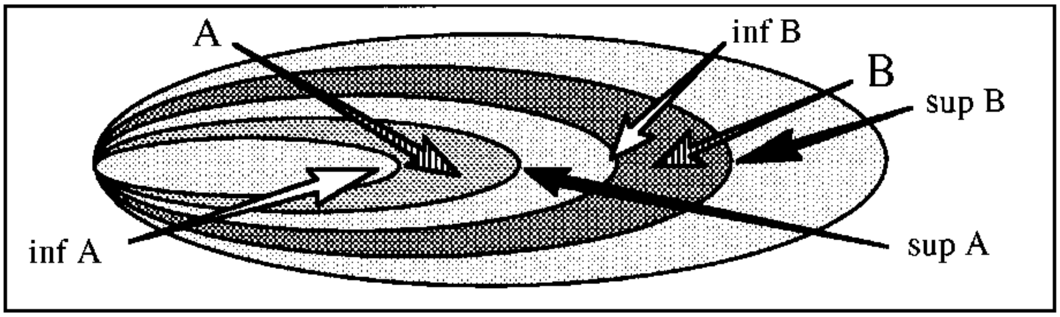

Figure 2: The distance of two disjoint segments in a totally ordered space.

Proof. Conditions for a space X (generally belonging to the set of parts of a space W) to be a topological space are three folds (Bourbaki, 1990): (i) Any union of sets belonging to X belongs to X. If A and B belong to X, then $\Delta(A, B) = \complement_{A \cup B}(A \cap B) \subseteq (A \cup B) \subset X$; (ii) Any finite intersection of sets belonging to X belongs to X. Let (A,B) $\in$ X. Since $\max \Delta(A, B) = (A \cup B)$ and $\min \Delta(A, B) = \emptyset$, and that $\emptyset \in X$ (Schwartz, 1991), then necessarily $(A, B) \in X$. The symmetric distance fulfills the triangular inequality, including in its generalized form, it is empty if A = B = ..., and it is always positive otherwise. It is symmetric for two sets and commutative for more than two sets. Its norm is provided by the following relation : $||\Delta(A)|| = \Delta(A, \emptyset)$.

Therefore, any topology provides the set-distance which can be called a topological distance and a topological space is always provided with a self mapping of any of its parts into any one metrics: thus any topological space is metrizable.

Reciprocally, given the set-distance, since it is constructed on the complementary of the intersection of sets in their union, it is compatible with existence of a topology. Thus, a topological space is always a "delta-metric" space.

Remark 3.2. Distance Δ(A,B) is a kind of an intrinsic case $[\Lambda_{(A,B)}(A,B)]$ of $\Lambda_E(A, B)$ while $\Lambda_E(A, B)$ is called a "separating distance". The separating distance also stands for a topological metrics. Hence, if a physical space is a topological space, it will always be measurable.

3.2. A corollary about intrinsic versus separating distances

3.2.1. Introduction

The previously proposed set distance as the symmetric difference between two or more sets is independent of any embedding space. It should thus be considered as an intrinsic one. However, the measure based on a path provided with a gauge pertaining to the common filter on A,B in E seeks for an identification of what is between A and B within E. Thus, a particular application can be raised :

3.2.2. Results

Proposition 3.2. Let spaces (A,B) $\subset$ E. Then a measure of the separating distance of A and B is defined if there exists a space X with nonempty intersections with E, A, B, such that, X belongs to the same filter $\mathcal{F}$ as A and B, and:

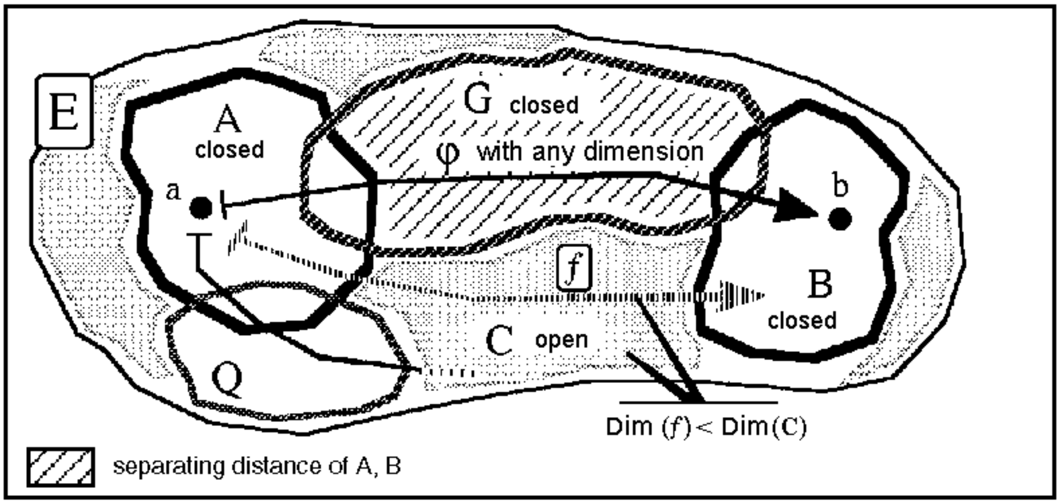

Figure 3: A path scanning the separating part between a closed space A and another closed space B must own a nonempty intersection with the objects situated between A and B. Sets A and G and sets B and G own their symmetric difference as an intrinsic set-distance. The intersection of these two measures is the separating distance.

Note: A set Q having empty intersection with either A or B (here with B) does not provide a separating distance since it does not necessarily fulfill the embedding space with connectivity. In this case, $\Delta(\mathrm{A, B, Q}) = \Delta(\mathrm{A, Q}) \cup \Delta(\mathrm{B}, \emptyset)$.

$$\Lambda_{\mathrm{E}}(\mathrm{A,B}) : \mathrm{E} \cap \{\Delta(\mathrm{A,X}) \cap \Delta(\mathrm{B,X})\} \tag{6}$$

Preliminary proof. Since filter $\mathcal{F}$ holds on E, and A,B$\in \mathcal{F}$ the three properties of a filter state the following (Bourbaki, 1990b, 1.36):

(i) $\mathrm{X} \in \mathcal{F}$ implies X must contain a set $\mathrm{G} \in \mathcal{F}$;
(ii) Since any finite intersection of sets of $\mathcal{F}$ must belong to $\mathcal{F}$, one has:

$$\mathrm{G} \in \{\mathrm{Ai, Aj} \in \mathcal{F},\ \mathrm{i} \neq \mathrm{j} :\ \mathrm{Ai} \cap \mathrm{Aj}\};$$

(iii) The empty part of X does not belong to $\mathcal{F}$. Therefore since $\mathrm{G} \in \mathrm{X}$ and $\mathrm{A} \cap \mathrm{B} = \emptyset$, then there must be $\mathrm{G} \in \mathrm{A} \cap \mathrm{X}$ and $\mathrm{G} \in \mathrm{B} \cap \mathrm{X}$, $\mathrm{G} \neq \emptyset$, which proves (4.1).

Hence, this example (Figure 3) further provides evidence that a definition is just a particular configuration of the intersection of two spaces of magmas of which one is the reasoning system (eventually a logic) and the other one is a probationary space.

3.2.3. Particular case: measuring open sets

As pointed above, even a continued path cannot in general scan a open component of the separating distance between two sets, since a path has in general no closed intersection with a open with same dimension. This is consistent with the exclusion of open adjoined intervals in the Borel measure. Hence, since a primary topology is a topology of open sets, a primary topological space cannot be a physically measurable space.

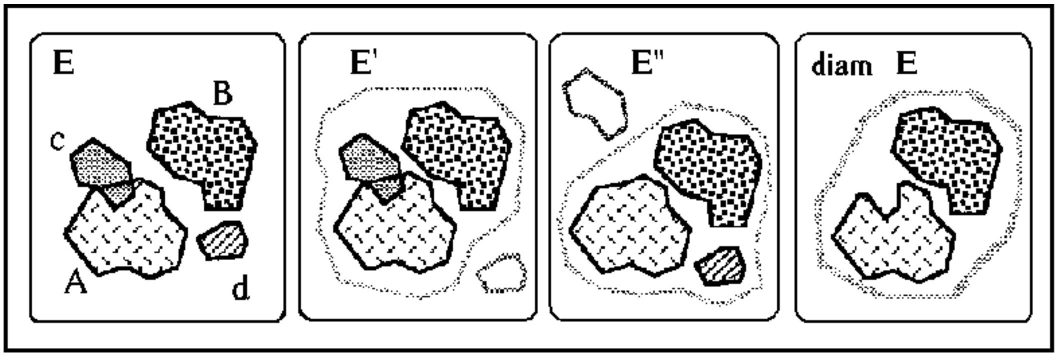

Figure 4: A preliminary approach of the diameter of an abstract set in a partly ordered space: this examples allows the $\sup(\mathrm{diam}\,\mathrm{E})$ as included in the complementary of c and d in E.

However, an intersection of a closed (C) with a path (φ) having a nonequal dimension than (C) owns a closed intersection with C provided this intersection is nonempty. This implies that the general conditions of filter membership is fulfilled (Figure 4).

Remark. An open 3-D universe would not be scanned by a 3-D probe. But in a closed Poincaré section, the topologies are distributed into closed parts and their complementaries as open subparts. Therefore, there may be open parts in our universe that would not be detectable by 3-D probes. This problem might be linked with the still pending problem of the missing dark matter (see also Arkani-Hamed, 2000).

3.2.4. An alternative perspective

Owing to the case in which there exists no intersection of spaces A and B with one of the other spaces contained in E, in order to define a common surrounding of A and B in an appropriate region of E, the following proposition would hold:

Proposition 3.3. A "surrounding distance" of spaces A and B in an embedding space E is given by the complementary of A and B in the interior of their common closure. This distance is noted $@_{\mathrm{E}}(\mathrm{A},\mathrm{B})$:

$$@_{\mathrm{E}}(\mathrm{A},\mathrm{B}) = \left[\{\overline{(\mathrm{A},\mathrm{B})^{\circ}}\}_{\mathrm{E}} \backslash (\mathrm{A}\cup\mathrm{B})\right] \supset \{\overline{\mathrm{A}}\} \cup \{\overline{\mathrm{B}}\}. \tag{7}$$

Corollary 3.3.1. A condition for the surrounding distance to be non-zero, is that A and B must be dense in E.

The closure of $\mathrm{A}\cup\mathrm{B}$ is different from, and contains, the union of the respective closures of A and B. This important property clearly delimitates a region of space E where any object may have to be scanned through a common gauge in order to allow a measure like $\Lambda_{\mathrm{E}}(\mathrm{A},\mathrm{B})$ to be allowed. Hence, the concept of surrounding distance is more general since it induces that of separating distance and belongs to a coarser topological filter.

3.3. About dimensionality studies

A collection of scientific observation through experimental devices produce images of some reality, and these images are further mapped into mental images into the experimentalist's brain (Bounias, 2000a). The information from the explored space thus stands like parts of an apparatus being spread on the worker's table, in view of a further reconstitution of the original object. We propose to call this situation an informational display, likely composed of elements with dimension lower than or equal to the dimension of the real object.

The next sections will thus deal with this particular problem.

3.3.1. Analysis of unordered versus ordered pairs in an abstract set

A robust definition of a ordered N-uple is given by the following:

Lemma 3.1 An expression noted (abc...z) is a ordered N-uple iff:

$$(\mathrm{abc...z}) = (\mathrm{x_1 x_2 x_3 ... x_n}) \;\Leftrightarrow\; \mathrm{a} = \mathrm{x_1},\ \mathrm{b} = \mathrm{x_2},\ \mathrm{c} = \mathrm{x_3},\ ...,\ \mathrm{z} = \mathrm{x_n}. \tag{8.1a}$$

In the construction of the set N of natural integers, Von Neumann provided an equipotent form using replicates of Ø:

$$0 = \emptyset,\ 1 = \{\emptyset\},\ 2 = \{\emptyset, \{\emptyset\}\},\ 3 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\},$$

$$4 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\},\ ...$$

A Von Neumann set is Mirimanoff-first kind since it is isomorphic with none of its parts. However this construction, associated with the application of Morgan's laws to (Ø), allowed the empty set to be attributed an infinite descent of infinite descents and thus to be classified as a member of the hypersets family (Bounias and Bonaly, 1997b).

Proposition 3.4. In a ordered pair {{x},{x,y}}, the paired part {x,y} is unordered.

A classical acceptance (Schwartz, 1991) states that {a,a} = {a}. This may introduce a confusion which can then be treated from relation (5.1a) :

$$(\mathrm{aa}) = (\mathrm{xy}) \;\Leftrightarrow\; \mathrm{a} = \mathrm{x},\ \mathrm{a} = \mathrm{y}. \tag{8.1b}$$

This just means that for any x,y, a pair (a,a) is not ordered. In effect, it becomes ordered upon rewriting: $\mathrm{A} = (\mathrm{a}, (\mathrm{a}, \mathrm{a}))$. Comparing (aa) and A through relation (8.1a), it comes: $(\mathrm{aa}) = \{\mathrm{a}, \{\mathrm{a}, \mathrm{a}\}\} \;\Leftrightarrow\; \mathrm{a} = \mathrm{a}$, and a={a,a}. (Q.E.D.)

Thus: {a,b} = {b,a}, while (ab) $\neq$ (ba) and $[\{\mathrm{a}, \mathrm{a}\} = \mathrm{a}] \neq [\{\mathrm{a}, (\mathrm{a})\} = (\mathrm{a}, \mathrm{a})]$ where (a) is a part, for which $\mathrm{a} \cup (\mathrm{a}) \neq \{\mathrm{a}\} \cup \{\mathrm{a}\}$. Consequently: (b,a) = (a,b) and (a,a) = !(a,a).

In a abstract set (in the sense of a set of any composition), one can find parts with nonempty intersections as well as parts or members with no common members. For instance, the following E and F are respectively without order and at least partly ordered:

$$\mathrm{E} = \{\mathrm{a},\ \mathrm{b},\ \{\mathrm{c}, \mathrm{d}\},\ \mathrm{G},\ \mathrm{X}\} \text{ with } \mathrm{a} \neq \mathrm{b},\ \mathrm{c} \neq \mathrm{d},\ \mathrm{G} \neq \mathrm{X}, \tag{8.2}$$

$$\mathrm{F} = \{\mathrm{a},\ \mathrm{b},\ \mathrm{c},\ \{\mathrm{a}, \mathrm{d}\},\ \{\mathrm{b}, \mathrm{c}, \mathrm{d}\},\ \{\mathrm{d},\ \{\mathrm{a}, \mathrm{e}, \mathrm{f}\}\}\}. \tag{8.3}$$

In particular cases, in a set like E or F, it may be that two singletons, e.g. {a}, {b}, though different, have the same weight with respect to a defined property. In such case one could write that there exists some (m) such that $\{a\} \equiv \{m\}$ and $\{b\} \equiv \{m\}$, so that $\{a, d, G\} \cap \{c, \{b, c\}, \} = (a \equiv b)$.

Proposition 3.5. An abstract set can be provided with at least two kinds of orders: one with respect to identification of a max and or a min, and one with respect to ordered N-uples. These two order relations become equivalent upon additional conditions on the nature of involved singletons.

(i) A Mirimanof set of the (2.2) type and derived forms can be provided a order. Let $E = \{e, \{f, G\}\}$. Then as seen above, there exists some m such that $\{e\} \equiv \{f\} \equiv \{m\}$, and E is similar to $E' \equiv \{m, \{m, G\}\}$. Then, for any $G \neq \emptyset$, $m \subset \{m, G\}$ and thus m is a minimal member of {m,G}. Pose $\{m, G\} = M$, then E′ becomes $E'' = \{m, M\}$ of which m is the minimal and M is the maximal of the set. Then, one can rewrite $\{m, \{m, G\}\}$ in the alternative form $\{m, (m, G)\}$ and $E = \{e, (f, G)\}$ since (f,G) is in some sort ordered. These notations will be respected below. Note that $\angle(f, G) \Rightarrow \exists\{f, \{G\}\}$ while the reciprocal is not necessarily true. The necessary condition is the following.

Suppose $\{f, \{G\}\} \Rightarrow \exists\{f, \{G\}\}$. Then, since $\{f, \{G\}\} = \{\{G\}, f\}$, one should be allowed to write $f \equiv G$ and $G \equiv f$, that is, there is neither minimal nor maximal in the considered part. Now, writing (f,G) imposes as a necessary condition that there exists some m and n such that $(f, G) \equiv \{m, \{m, n\}\} = \{m, \{n, m\}\}$. This can then be (but only speculatively) turned into a *virtual* (m,(m,n)). When comparing two sets, parts ordered by this way will have to be compared two by two: $\{e, \{f, G\}\} = \{m, (n, Z)\} \Rightarrow e = m$, and since $m \neq M$: $f = n$, $G = Z$ (otherwise $\{f, g\} = (n, z)$ could give alternatively $f = z$, $n = g$).

This drives the problem to the identification of orders in the set of parts of a set, as compared with components of a simplex set (Table 1).

(ii) A set can be ordered through rearrangement of its exact members and singletons in a way permitted by the structure of the set of parts of itself. In effect, existence of a set axiomatically provides existence to the set of its parts (Bourbaki, 1990a, p. 30) though an axiom of availability has been shown to be required for disposal of the successive sets of parts of sets of parts (Bounias, 2001).

Now, how to identify preordered pairs in a set? The set of parts $\mathcal{P}(E)$ of a set E provides the various ways members of this set can be gathered in subsets, still not ordered. Therefore, an analysis of the members of a set can involve a rearrangements of the singletons contained in the set in a way permitted by the arrangements allowed by $\mathcal{P}(E)$, including when a same singleton is present several times in the set. Then, this may let emerge configurations that can be identified with a structure of ordered N-uples.

Remark 3.3. In N-uples as well as in the set of parts, singletons gathered in any subpart are unordered. Call (E′) a set reordered by using $\mathcal{P}(E)$. Then: $E' \cap \mathcal{P}(E) = \mathbf{Partition}(E)$.

Application to simplicial structures

Table 1 illustrates for the first four simplexes, the comparison of the ordered pairs of the simplexes with the remaining part of the set of parts, which is constituted of

$(2^N - N - 1)$ unordered nonempty members for a simplex of N vertices.

Table 1. Sets and simplexes: N and D are the numbers of vertices and the maximum dimension, respectively. $\mathcal{P}$(En) denotes the set of nonempty parts of each n considered. Each class of subsets of k members appears in C_N^k forms. One of each is used in a ordered n-uple, so that $C_N^k - 1$ are remaining. Finally, $\sum_{n=1\to N}(C_n^k - 1) + N = 2^N$.

Sets	Vertices	Ordered pairing (N)	Dimensionality	Remaining of $\mathcal{P}$(En) $(2^N - N - 1)$	$(2^N - 1)$
E1 = {x1}	N=1	{x1}	D=0	none	1
E2 = {x1, x2}	N=2	{x1},(x1,x2) = (x1x2)	D=1	{x2}	3
E3={x1,x2,x3}	N=3	{x1},(x1,x2), (x1,x2,x3) = (x1x2x3)	D=2	{x2},{x3},{x1,x3},(x2,x3)	7
E4={x1,x2,x3,x4}	N=4	{x1},(x1,x2), (x1,x2,x3), (x1,x2,x3,x4) = (x1x2x3x4)	D=3	{x2},{x3},{x4}, {x1,x3},{x1,x4},{x2,x3}, {x2,x4}, {x3,x4}, {x1,x2,x4}, {x1,x3,x4}, {x2,x3,x4}	15

The remaining parts can in turn be arranged into appearing ordered subparts. Hence for E3: $\{\{x2\},\{x2,x3\}\}$ = (x1x3) and the *"in fine"* ordered $\{\{x3\},\{x1,x3\}\}$ = (x3xl) since {x1,x3}=(x3,x1). For E4, one gets $\{\{x2\},\{x2,x3\},\{x2,x3,x4\}\}$ = (x2x3x4), $\{\{x3\},\{x3,x4\}\}$ = (x3x4), $\{\{x1,x3\},\{x1,x3,x4\}\}$ = ((x1, x3)x4), and the *"in fine"* ordered: $\{\{x4\},\{x1,x4\}\}$ = (x4x1), $\{\{x2,x4\},\{x1,x2,x4\}\}$ = ((x2, x4)x1).

This further justifies that for a set like (En), the set of its (2^n) parts has dimension $D \leq D(E^n)$).

In fine, a fully informative measure should provide a picture of a space allowing its set component to be depicted in terms of ordered N-uples ($N = 0 \to n$, $n \subset \mathbf{N}$).

These considerations are intended to apply when a batch of observational data and measurement are displayed in a scattered form on the laboratory table of a scientist, and must then be reconstructed in a way most likely representative of a previously unknown reality.

3.3.3. Identification of a "scanning" measure on an abstract set

Several kinds of measure of a set, including various forms of its diameter infer from section 2.1 above.

Proposition 3.6. A set can be scanned by the composition of a identity function with a difference function. Let E = {a,b,c,...} a set having N members.

(i) An identity function Id maps any members of E into itself: $\forall x \in E$, $Id(x) = x$.

Thus, $\angle$(a, or b, or c, ...) this provides one and only one response when applied to E.

(ii) A difference function is f such that: $\forall x \in E$, $f^n(x) \neq x$.

The exploration function is a self-map $\mathbf{M}$ of E: $\mathbf{M} : E \mapsto E$, $\mathbf{M} = Id \perp f$:

$$\angle x \in E,\ \mathbf{M}(x) = f(Id(x)),\ \forall n\ :\ \mathbf{M}^n(x) = f^n(Id(x)) \neq x. \tag{9.1}$$

<u>Proof.</u> (a) Suppose $\mathbf{M} = Id(x)$; then, each trying maps a member of E to a fixed point and there is no possible scanning of E. (b) Suppose one poses just $f(Id(x)) \neq x$; then, given $f(Id(x)) \neq x$, say $f^1(Id(x) = y$, since $y \neq x$, then one may have again $f^2(Id(x)) = x$. Therefore, there can be a loop without further scanning of E, with probability $(N-l)^1$. (c) Suppose one poses $M = f$, such that $f(x) \neq x$. Then, since $f^0(x) \neq x$, there can be no start of the scanning process. If in contrast one poses as a modification of the function f that $\angle x$, $f^0(x) = x$, this again stops the exploration process, since then $f^0 \in Id$. This has been shown to provide a minimal indecidability case (Bounias, 2001).

The sequence of functions $M^n(x) = f^n(Id(x))) \neq x$, $\forall n$, is thus necessary and sufficient to provide a measure of E which scans N-l members of E. The sequence stops at the N^{th} iterate if in addition:

$$f^n(Id(x)) \neq \{f^i(Id(x))\}_{\forall i \in [1,N]}. \tag{9.2}$$

The described sequence represents an example of a path as described above in more general terms.

Now, some preliminary kinds of diameters can be tentatively deduced for the general case of a set E in which neither a complete structure nor a total order can be seen.

3.3.4. Tentative evaluations of the size of sets with ill-defined structure and order

Since in this case any two members of E are of similar weight, regarding the definition of a diameter (4.3d), it can be proposed the following definition.

Definition 3.1a. Given a non-ordered set E, Id the identity self map of E, and f the difference self map of E, a kind of diameter is given by the following relation:

$$\mathrm{diam} f(E) \approx \{(x,y) \in E : \max f^i(Id(x)) \cap \max f^i(Id(y)\}. \tag{10.1}$$

This gives a (N-2) members parameter.

Subdefinition 3.1b. If E is well-ordered, i.e. at least one, e.g. the lower boundary can be identified among members of E, such as a singleton $\{m\}$, then an alternative form can be written. Since in this case, the set E can be represented by two members: $E = \{m, Z\}$ with $Z = \complement_E(m)$ the complementary of m in E, relation (4.3d) results in a measure $M_{H\Delta}$:

$$M_{H\Delta}(E) = \{m = \min(E),\ \Delta(m, Z)\} \tag{10.2}$$

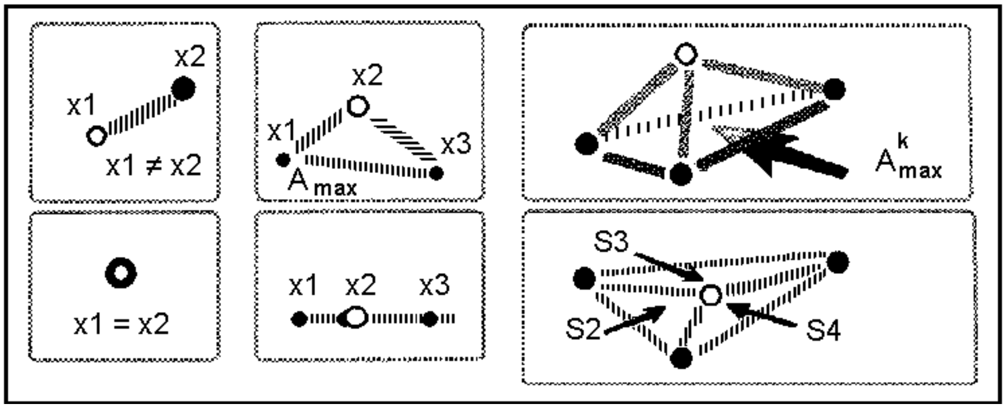

Figure 5: The first three steps of the N-angular strict inequality for the assessment of the dimensionality of a simplex. In the lower right picture, the larger side standing for A^{k}_{max} is S1 such that in a 2-D space one has exactly: $S1 = S2 + S3 + S4$.

with Δ the symmetric difference.

This gives a (N-1) member parameter which in turn provides a derived kind of diameter $\mathrm{diam}_{H\Delta}$ by repeating the measure for two members m' and m'' as on Figure 4. Let $E' = \{m', \complement_E(m')\}$, and $E'' : \{\complement_E(m'')\}$, then (Figure 5):

$$\mathrm{diam}(E) \subseteq \{m', m'' \in E : \max \Delta(E', E'')\}. \tag{10.3}$$

Subdefinition 3.1c. If both upper and lower boundaries can be identified, i.e. the set E is totally ordered, then the distance separating two segments A and B of E is:

$$(A, B) \subset E,\ \mathrm{dist}_E(A, B) = \{\mathrm{dist}(\inf A,\ \inf B) \cap \mathrm{dist}(\sup A,\ \sup B)\} \tag{10.4}$$

where a set distance Δ is again provided by the symmetric difference or by a n-D Borel measure.

A separating distance is a extrinsic form of the set-distance as an intrinsic form.

A diameter is evaluated on E as the following limit:

$$\begin{aligned} \mathrm{diam}(E) = \{&\inf A \to \inf E,\ \sup A \to \min E, \\ &\inf B \to \max E,\ \sup B \to \sup E \mid \lim \mathrm{dist}_E(A, B)\} \end{aligned} \tag{10.5}$$

These preliminary approaches allow the measure of the size of tessellating balls as well as that of tessellated spaces, with reference to the calculation of their dimension through relations (2.1)-(2.2) and (11.1)-(11.2) derived below. A diagonal-like part of an abstract space can be identified with and logically derived as, a diameter.

Remark 3.4. If measure obtained each time from a system, this means that no absolutely empty part is present as an adjoined segment on the trajectory of the exploring path. Thus, no space accessible to some sort of measure is strictly empty in a both mathematical and physical sense, which supports the validity of the quest of quantum mechanics for a structure of the void.

3.3.5. The dimension of an abstract space: tessellatting with simplex k-faces

A major goal in physical exploration will be to discern among detected objects which ones are equivalent with abstract ordered N-uples within their embedding space.

A first of further coming problems is that in a space composed of members identified with such abstract components, it may not be found tessellatting balls all having identical diameter. Also a ball with two members would have no such diameter as defined in (10.1) or (10.4). Thus a measure should be used as a probe for the evaluation of the coefficient of size ratio (p) needed for the calculation of a dimension.

Some preliminary proposed solutions hold on the following principles.

(i). A 3-object has dimension 2 iff given A^1_{max} its longer side fulfills the condition, for the triangular strict inequality, where M denotes an appropriate measure:

$$M(A^1_{max}) < M(A^1_2) + M(A^1_3). \tag{11.1a}$$

Similarly, this condition can be extended to higher dimensions (Figure 5):

$$M(A^2_{max}) < M(A^2_2) + M(A^2_4) \tag{11.1b}$$

Then, more generally for a space X being a N object:

$$M(A^k_{max}) < \bigcup_{i=1}^{N-1} \{ M(A^k_i)\} \tag{11.1c}$$

with N = number of vertices, i.e. eventually of members in X, $k = (d-1) = N-2$, and A^k_{max} the k-face with maximum size in X. This supports a former proposition (Bounias, 2001).

Remark 3.5. One should note that, according to relation (11.1c), for N = 2 (a "2-object"), $X = \{x1, x2\}$ has dimension 1 iff $x1 < (x1 + x2)$, that is iff $x1 < x2$ (Figure 5). This qualifies the lower state of an existing space X^1.

(ii). Let now space X be decomposed into the union of balls represented by D-faces A^D proved to have dimension $\mathrm{Dim}\,(A^D) = D$ by relation (11.1c) and size $M(A^1)$ for a 1-face. Such a D-face is a D-simplex Sj whose size, as a ball, is evaluated by $M(A^1_{max})^D = Sj^D$. Let $\mathcal{N}$ is the number of such balls that can be filled in a space H, so that

$$\bigcup_{i=1}^{\mathcal{N}} \{Sj^D\} \subseteq (H \approx L^d_{max}) \tag{11.2a}$$

with H being identified with a ball whose size would be evaluated by L^d, L is the size of a 1-face of H, and d the dimension of H. Then, if $\forall\, Sj,\ Sj \approx So$, the dimension of H is:

$$\mathrm{Dim}(H) \approx (D \cdot \mathrm{Log}\, So + \mathrm{Log}\,\mathcal{N}) \,/\, \mathrm{Log}\, L^1_{max}. \tag{11.2b}$$

Remark. Relation (11.2b) stands for a kind of interior measure in the Jordan's sense. In contrast, if one poses that the reunion of balls covers the space H, then,

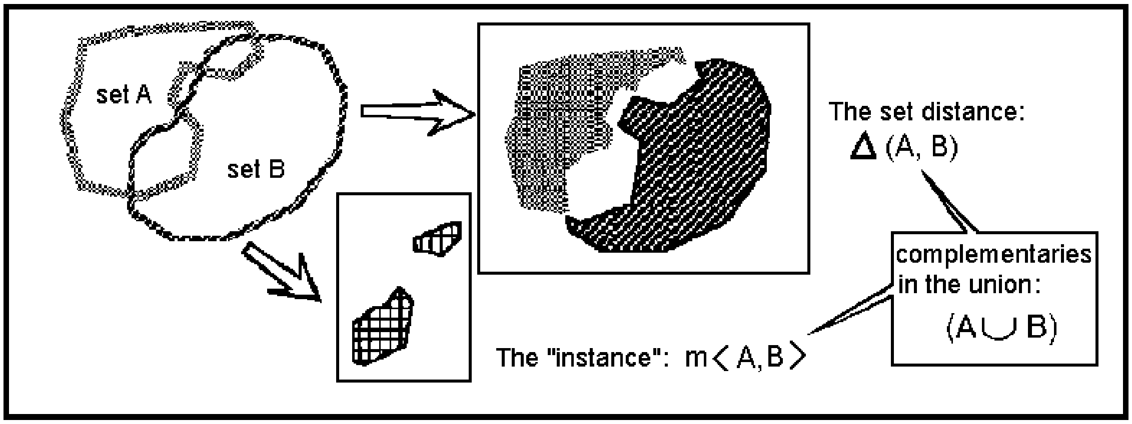

Figure 6: The symmetric difference (Δ) between two or more sets provides a kind of distance which is consistent with a topology. This distance therefore stands for a topological distance and provides topological spaces with a general form of metrics. By this way, any topological space may be treated like a (delta-) metric space.

Dim(H) rather represents the capacity dimension, which remains an evaluation of a fractal property.

Definition 3.2b. A n-(simplex-ball) is the topological unit ball circumscribed to a n-simplex.

(iii). Extension to a ill-defined space E.

The problem consists in identifying first a 1-face component of E from k-faces ($k > l$), which implies to identify ordered N-uples. Then, components of E, whatever their nature, should be analogically decomposed into appropriate simplexes. The number of these simplexes will then be evaluated in E, and relations (11.2) will be finally applied.

Remark. There is no other condition on simplexes Sj than the need that one face is maximal: this is the face on which the others will be projected, so that the generalized inequality will be applied (Figure 6).

Depending on $\mathcal{N}$, noninteger d(H) can be obtained for a fractal or a fractal-like H. Some applications of the above protocol will be illustrated in further communications.

Lemma 3.2. A singleton or member (a) is putatively available in the form of a self-similar ordered N-uple: $\{a\} = \{a \cap a \cap \ldots \cap a\}_h \neq \{(a, (a), \ldots\}$.

In effect, the set theory axiom of the reunion states that $A \cup A = A$ as well as the axiom of the intersection states that $A \cap A = A$.

Corollary 3.2. In the analysis of a abstract space $H = L^x$, of which dimension x is unknown, the identification of which members can be identified with N-uples supposedly coming from a putative Cartesian product of members of H, e.g.: $G \subset H$: $\angle a \in G$, $(aaa...a)_h \in G^h$, is allowed by an anticipatory process.

Preliminary proof. At time (t) of the analysis of the formal system involved, there is no recurrent function that can "imagine" the existence of an abstract component not existing in the original data and parameters, and not directly inferring from a computation of these data and parameters. Devising $(\mathrm{aaa...a})_{\mathrm{h}} \in \mathrm{G}^{\mathrm{h}}$, implies making a mental image at $(\mathrm{t} + \mathrm{te})$ and further confronting the behavior of the system with "anticipatively recurrent" images succeeding to those at t, that is computed from $(\mathrm{t}, \mathrm{t} + \mathrm{e})$ conditions within (t+1, t+2, ..., up to $\mathrm{t} + \mathrm{e} - 1$).

Let $\mathrm{E} = \{\mathrm{a}, \{\mathrm{b}, \{, \mathrm{B}, \mathrm{C}\}\}\}$; then one anticipates on the putative existence of a unordered pair $\{\mathrm{a}, \mathrm{a}\} = \mathrm{a}$ in a former writing of E.

Therefore, $(\mathrm{a}) = \big\{\{\mathrm{a}, \{\mathrm{a}, \mathrm{a}\}\} | \{\mathrm{a}, \mathrm{a}\} = \{\mathrm{a}\}\big\}$. The presence of singletons can be identified with a putative former reduction of ordered parts owning the same nucleus (a). A nucleus thus appears as a ordered form of a singleton, that is the only case where a ordered form is identical with a unordered one.

Application 3.2. Given $(\mathrm{E}^{\mathrm{e}}_{\mathrm{n}})$ composed of N-uples denoted $(\Pi^{\mathrm{e}}_{\mathrm{i}})$ in their ordered state, and $\mathrm{K}^{\mathrm{e}}_{\mathrm{i}}$, $\mathrm{K}^{\mathrm{e}}_{\mathrm{j}}$, ... , in their unordered acceptation: an approach of the identification of the diagonal, and thus of a further measure of the respective diameters of $(\mathrm{E}^{\mathrm{e}}_{\mathrm{n}})$ and $\Pi^{\mathrm{e}}_{\mathrm{i}})$ is given by the following propositions:

$$\mathrm{diag}\,(\mathrm{E}^{\mathrm{e}}_{\mathrm{n}}) \subseteq \min \cup \{(\mathrm{K}^{\mathrm{e}}_{\mathrm{i}}, \mathrm{K}^{\mathrm{e}}_{\mathrm{j}}) \in \mathrm{E}^{\mathrm{e}}_{\mathrm{n}} : \Delta(\mathrm{K}^{\mathrm{e}}_{\mathrm{i}}, \mathrm{K}^{\mathrm{e}}_{\mathrm{j}})_{\mathrm{i} \neq \mathrm{j}}\} \tag{11.3a}$$

$$\mathrm{diag}(\Pi^{\mathrm{e}}_{\mathrm{i}}) \subseteq \max \cup \{(\mathrm{K}^{\mathrm{e}}_{\mathrm{i}}, \mathrm{K}^{\mathrm{e}}_{\mathrm{j}}) \in \mathrm{E}^{\mathrm{e}}_{\mathrm{n}} : \Delta(\mathrm{K}^{\mathrm{e}}_{\mathrm{i}}, \mathrm{K}^{\mathrm{e}}_{\mathrm{j}})_{\mathrm{i} \neq \mathrm{j}}\} \tag{11.3b}$$

An example is given in Appendix as a tentative application case. Regarding just a Cartesian product, the set of parts with empty or minimal intersections stands for the diagonal and the diagonal of a polygon stands for a topological diameter.

4. Physical inferences

4.1. Defining a probationary space

A probationary space (Bounias, 2001) is defined as a space fulfilling exactly the conditions required for a property to hold, in terms of:

(i) identification of set components;
(ii) identification of combinations of rules
(iii) identification of the reasoning system.

All these components are necessary to provide the whole system with decidability (Bounias, 2001). Since lack of mathematical decidability inevitably flaws also any physical model derived from a mathematical background, our aim is to access as close as possible to these imposed conditions in a description of both a possible and a knowable universe, in order to refine in consequence some current physical postulates.

The above considerations have raised a set of conditions needed for some knowledge to be gettable about a previously unknown or ill-known space, upon pathwise exploration from a perceiving system A to a target space B within an embedding space B.

In the absence of preliminary postulate about existence of so-called "matter" and related concepts, it has been demonstrated that the existence of the empty set is a necessary and sufficient condition for the existence of abstract mathematical spaces (W^n) endowed with topological dimensions (n) as great as needed (Bounias and Bonaly, 1997). Hence, the empty set appears as a set without members though containing empty parts. The reasoning intermediately proved that the empty set owns the properties of a nonwellfounded set and exhibit (i) self-similarity at all scales and (ii) nowhere derivability, that is: two characteristics of fractal structures. These properties have been shown to clear up some antinomies remained unaddressed about the empty set properties.

These findings will now be shown to provide additional features of physical interest. In first, the empty set can provide at least an intellectual support to existence of some sort of space.

4.2. The founding element

It is generally assumed in textbooks of mathematics (see for instance Schwartz, 1991, p. 24), that some set does exist. This strong postulate has been reduced to a weaker form reduced to the axiom of the existence of the empty set (Bounias and Bonaly, 1997). It has been shown that providing the empty set ($\emptyset$) with ($\in, \subset$) as the combination rules (that is also with the property of complementarity ($\complement$)) results in the definition of a magma allowing a consistent application of the first Morgan's law without violating the axiom of foundation iff the empty set is seen as a hyperset, that is a nonwellfounded set (Aczel, 1987, Barwise, 1991). A further support of this conclusion emerges from the fact that several paradoxes or inconsistencies about the empty set properties are solved (Bounias and Bonaly, 1997). These preliminaries now drive us to the formulation of the following theorem which will be established using several Lemmas.

4.3. The founding lattice

Theorem 4.1. The magma $\emptyset^{\emptyset} = \{\emptyset, \complement\}$ constructed with the empty hyperset and the axiom of availability is a fractal lattice.

Remark. The notation $\emptyset^{\emptyset}$ has been preferred to -say- ($E_{\emptyset}$) or others. In effect, in Bourbaki notation, E^F means the space of functions of set F in set E. Therefore, writing ($\emptyset^{\emptyset}$) denotes that the magma reflects the set of all self-mappings of ($\emptyset$), which emphasizes the forthcoming results.

Lemma 4.1. The space constructed with the empty set cells of $E\emptyset$ is a Boolean lattice.

Proof. (i) Let $\cup(\emptyset) = S$ denote a simple partition of ($\emptyset$). Suppose that there exists an object (ε) included in a part of S, then necessarily $(\varepsilon) = \emptyset$ and its belongs to the partition.

(ii) Let $P = \{\emptyset, \emptyset\}$ denote a part bounded by $\sup P = S$ and $\inf P = \{\emptyset\}$. The combination rules $\cup$ and $\cap$ provided with commutativity, associativity and absorption are holding. In effect: $\emptyset \cup \emptyset = \emptyset$, $\emptyset \cap \emptyset = \emptyset$ and thus necessarily $\emptyset \cup (\emptyset \cap \emptyset) = \emptyset$, $\emptyset \cap (\emptyset \cup \emptyset) = \emptyset$. Thus, space $\{(P(\emptyset), (\cup, \cap)\}$ is a lattice.

The null member is Ø and the universal member is $2^{\emptyset}$ that should be denoted by $\aleph_{\emptyset}$. Since in addition, by founding property $\complement_{\emptyset}(\emptyset) = \emptyset$, and the space of (Ø) is distributive, then S(Ø) is a Boolean lattice. (Q.E.D.)

Lemma 4.2. S(Ø) is provided with a topology of discrete space.

<u>Proof.</u> (i) The lattice S(Ø) owns a topology. In effect, it is stable upon union and finite intersection, and its contains (Ø).

(ii) Let S(Ø) denote a set of closed units. Two units $\emptyset_1$, $\emptyset_2$ separated by a unit $\emptyset_3$ compose a part $\{\emptyset 1, \emptyset_2, \emptyset_3\}$. Then, owing to the fact that the complementary of a closed is a open: $\complement_{\{\emptyset_1,\emptyset_2,\emptyset 3\}}\{\emptyset_1, \emptyset_3\} = \emptyset_2$, and $\emptyset_2$ is open. Thus, by recurrence, $\{\emptyset 1, \emptyset_3\}$ are surrounded by open]Ø[and in parts of these open, there exists distinct neigborhoods for $(\emptyset_1)$ and $(\emptyset_3)$. The space S(Ø) is therefore Hausdorff separated. Units (Ø) formed with parts thus constitute a topology $(\mathcal{T}_{\emptyset})$ of discrete space. Indeed, it also contains the discrete topology $(\emptyset^{\emptyset}, (\emptyset))$, which is the coarse one and is of much less mathematical interest.

Lemma 4.3. The magma of empty hyperset is endowed with self-similar ratios.

The Von Neumann notation associated with the axiom of availability, applying on (Ø), provide existence of sets $(N^{\emptyset})$ and $(Q^{\emptyset})$ equipotent to the natural and the rational numbers (Bounias and Bonaly, 1997). Sets Q and N can thus be used for the purpose of a proof. Consider a Cartesian product En × En of a section of $(Q^{\emptyset})$ of n integers. The amplitude of the available intervals range from 0 to n, with two particular cases: interval [0,1] and any of the minimal intervals [1/n − l, 1/n]. Consider now the open section]0,1[: it is an empty interval, noted $\emptyset_1$. Similarly, note $\emptyset_{\min}$ =]0, l/n(n−l)[. Since interval [0, l/n(n−l)] is contained in [0,1], it follows that $\emptyset_{\min} \subset \emptyset_1$. Since empty sets constitute the founding cells of the lattice S(Ø) proved in Lemma 4.1, the lattice is tessellatted with cells (or balls) with homothetic-like ratios of at least r = n(n − l). The absence of unfilled areas will be further supported in Part 2 of this study by introduction of the "set with no parts".

Definition 4.1. Such a lattice of tessellation balls will be called a "tessellattice".

Lemma 3.3.4. The magma of empty hyperset is a fractal tessellattice.

<u>Proof.</u>

(i) From relations (2.3) one can write $(\emptyset) \cup (\emptyset) = (\emptyset, \emptyset) = (\emptyset)$.
(ii) It is straightforward that $(\emptyset) \cap (\emptyset) = (\emptyset)$.
(iii) Last, the magma $(\emptyset^{\emptyset}) = \{\emptyset, \complement\}$ represents the generator of the final structure, since (Ø) acts as the "initiator polygon", and complementarity as the rule of construction. These three properties stand for the major features which characterize a fractal object (James and James, 1992).

Finally, the axiom of the existence of the empty set, added with the axiom of availability in turn provide existence to a lattice S(Ø) that constitutes a discrete fractal Hausdorff space, and the proof is complete.

4.4. Existence and nature of spacetime

Lemma 4.5. A lattice of empty sets can provide existence to a at least physical-like space.

Proof. Let Ø denote the empty set as a case of the whole structure, and {Ø} denotes some of its parts. It has been shown that the set of parts of Ø contains parts equipotent to sets of integers, of rational and of real numbers, and owns the power of continuum (Bounias and Bonaly, 1994,1997). Then, looking at the inferring spaces $(\mathrm{W}^{\mathrm{n}})$, $(\mathrm{W}^{\mathrm{m}})$, ... thus formed, it has been proved (Bounias and Bonaly, 1994) that the intersections of such spaces having nonequal dimensions give raise to spaces containing all their accumulation points and thus forming closed sets. Hence:

$$\{(\mathrm{W}^{\mathrm{n}}) \cap (\mathrm{W}^{\mathrm{m}})\}_{\mathrm{m}>\mathrm{n}} = (\Theta^{\mathrm{n}}) \text{ is closed space.} \tag{12}$$

These spaces provide collections of discrete manifolds whose interior is endowed with the power of continuum. Consider a particular case (Θ^4) and the set of its parts $\mathcal{P}(\Theta^{\mathrm{n}})$; then any of intersections of subspaces $(\mathrm{E}^{\mathrm{d}})_{\mathrm{d}<4}$ provides a d-space in which the Jordan-Veblen theorem allows closed members to get the status of both observable objects and perceiving objects (Bounias and Bonaly, 1997b). This stands for observability, which is a condition for a space to be in some sort observable, that is physical-like (Bonaly's conjecture, 1992).

Finally, in any (Θ^4) space, the ordered sequences of closed intersections $\{(\mathrm{E}^{\mathrm{d}})_{\mathrm{d}<4}\}$, with respect to mappings of members of $\{(\mathrm{E}^{\mathrm{d}})_{\mathrm{d}<4}\}_{\mathrm{i}}$ into $\{(\mathrm{E}^{\mathrm{d}})_{\mathrm{d}<4}\}_{\mathrm{j}}$, provides an orientation accounting for the physical arrow of time (Bounias, 2000a), in turn embedding an irreversible arrow of biological time (Bounias, 2000b). Thus the following proposition:

Proposition 4.1. A manifold of potential physical universes is provided by the (Θ^{n}) category of closed spaces.

Our spacetime is one of the mathematically optimum ones, together with the alternative series of $\{(\mathrm{W}^3) \cap (\mathrm{W}^{\mathrm{m}})\}_{\mathrm{in}>3}$. Higher spacetimes $(\Theta^{\mathrm{n}})_{\mathrm{n}} > 3$ could exist as well.

Now it will be briefly examined some new structures which can pertain to a topological space as described above, and deserves specific attention.

5. Towards "fuzzy dimension" and "beaver spaces"

5.1. Introduction

The exploration of nature raises the existence of strange objects, such as living organisms, whose anatomy suggests that mathematical objects having adjoined parts with each having different dimensions could exist. This will then introduce the idea that the dimensions of some objects may even not be completely established. The existence of such strange objects implies that appropriate tools should be prepared for their eventual study. This is sketched here and will be matter of further developments.

5.2. From hairy spaces to "beaver spaces"

A "hairy space" is a $(n > 3)$-ball having 1-D lines (also called "hair" of "grass") planted on it (Berger, 1990). It is interesting that for such spaces ($n > 3$ only), the volume and the area do not change with the insertion of 1-spaces on them.

Let a simplex S_n^n: the last segment allowing set E_{n+1} to be completed from E_n of S^{n-1} is (x_n, x_{n+}). Consider the simplex F^{n-1} such that one of its facets A^{n-2} has its last segment $(y_{n-1}, y_{n-2}) \equiv (x_n, x_{n+1})$. Repeat this operation for descending values: one gets a space having n- and $(n-l)$-adjoined parts with their intersections having lower dimension: $\mathrm{Dim}\{S^n \cap S^{n-1}\} \leq (n-2)$. With respect to a beaver, having a spherical body, with a flat tail surrounded with hair, these spaces will be denoted as "beaver spaces". Other possibilities include not ordered appositions of parts with various dimensions.

The existence of Beaver spaces implies some specific adjustment of the methods used for their scanning. In this respect, it should be recalled that a new mode of assessment of coordinates has been formerly proposed (Bounias and Bonaly, 1996): it consists in studying the intersection of the unknown space with a probe composed with an ordered sequence of topological balls of decreasing dimensions, down to a point ($D = 0$). This process has been shown to wear the advantage of being able to define coordinates even in a fractal space. This may be of particular interest with components provided by and embedded in the lattice $S(\emptyset)$.

5.3. "Fuzzy dimensioned" spaces

Proposition 5.1. There exist spaces with fuzzy dimensions.

<u>Proof.</u> Consider a simplex $S_n^n = \{(E_{n+}), (\perp n)\}$, and its two characteristic structures, namely $\mathbf{L}_n^1 = \sum_{i=1\to n} \left(\mathrm{dist}(x_{i-1}, x_i)\right)$ and $¥_{n+1}^1 = \sum_{i=1\to n} \left(\mathrm{dist}(x_{i+1}, x_i)\right)$. Then, a condition for the assessment of simplicial dimensionality is given by the structure defining a simplicial space (Bounias, 2001) in consistency with a $(1-D)$-probe for a $(D > l)$-space:

$$(\perp n) = (¥_{n+1}^1 > k(n, d) \cdot \mathbf{L}_n^1 \Leftrightarrow d > n). \tag{13}$$

Let the last segment of the simplex (x_n, x_{n+1}) be such that, in consistency with Zadeh and Kacprzyk (1992), one has: $\mathrm{dist}(x_n, x_{n+1}) \in [0, 1]$. Then, the expression of $¥_{n+1}^1$ reaches a value fuzzily situated between the assessment of $D = n - 1$ and $D = n$ at least for $n < 4$. Thus, the simplicial set E_{n+1} is of the fuzzy type and the simplex is a fuzzy simplex.

Such a space will therefore be said as having a "fuzzy dimension". This provides a particular extension of the concept of fuzzy set into that of "fuzzy space of magma", since the set is unchanged while it is the rule that provides the magma with a fuzzy structure.

These problems will be the matter of further developments, since they belong to the kind of situations that could really be encountered during the exploration of universe, and as pointed by Klir and Wierman (1999): "Knowledge about the outcome of an uncertain event gives the possessor an advantage".

6. Discussion

6.1. Epistemological assessment

It has been stressed in the introduction that computable systems can in principle not be self-evaluated, due to incompleteness and indecidability theorems. It will be shown here on a simple example why this does not apply to the self-evaluation of our universe structure by one of its components, when this component is a living subpart provided with conscious perception functions, that is a brain.

Theorem 6.1. A world containing subparts endowed with consciously perceptive brains is a self-evaluable system.

The proof will be founded on two main Lemmas.

Lemma 6.1. There exists a minimal indecidable system (Bounias, 2001).

Proof. The former theorem of Godel provided the most sophisticated proof, while the systems devised by Chaitin described the most giant examples of known indecidable systems. In contrast, a recurrent reasoning would state that there should exist most simpler cases. Consider the set $\mathrm{E} = \{(\mathrm{A})\}$ where (A) is a part, eventually composed of nuclei or singletons. Try an exploration of E by itself: without prior knowledge of the inside structure of E, the exploration function $f^{\mathrm{n}}(\mathrm{Id}(\mathrm{x}))$ described in relations (2.3.1–2.3.5). Applying this function to E, one gets $\mathrm{Id}(\mathrm{A}) = \mathrm{A}$ and $f^{\mathrm{n}}(\mathrm{A}) \neq \mathrm{A}$ does not exists from the definitions (this applies to $\mathrm{A} = \{\mathrm{a}\}$ a singleton, as well). Hence, the system returns no result: not a zero result, but litterally no answer. Now, let A be composed of singletons, the same procedure must be applied to each detected singleton, with the same failure. Finally, let A be ($\emptyset\varnothing$): for any singleton $\{\emptyset\}$, one will get $\mathrm{Id}(\emptyset) = \emptyset$ and $f^{\mathrm{n}}(\emptyset)$ again returns no response. Hence, the lattice S(0) is not self-evaluable as it stands at this stage. Since $\{\emptyset\}$ is the minimum of any nonempty set, the system stands for a minimum of an indecidable case of the classical set theory (though a new component will be considered in Part 2).

Lemma 6.2. A consciously perceptive biological brain is endowed with anticipatory properties (Bounias and Bonaly, 2001).

The function of conscious perception has been shown to infer from the same conditions as those allowing a physical universe to exist from abstract mathematical spaces: A path connecting a Jordan's point of an outside closed B to the inside of another closed A is prolonged in a biological brain into a sequence of neuronal configurations which converges to fixed points. These fixed points stand for mental images (Bounias and Bonaly, 1996; Bounias, 2000). The sequence of mental images owns fractal properties which can provide additional help to the construction by the brain of mental images of an expected future state. These mental images will in turn be used as a guide for the adjustment of further actions to the expected goal (Bounias, 2000), which is also a way by which the organism returns molecular information to the brain, making unconscious (autonomous) mental images which are used in turn for the control of the homeostasy of the organism.

Now, consider a space $\mathrm{F} = \{(\mathrm{A}), (\mathrm{B})\}$ where (A) is a living system endowed with anticipatory properties. Then, A is able to analyze some of its components (a_1, a_2, ...) through a way similar to the exploration function. By an anticipatory process, it is able to construct the power set of parts $\mathcal{P}^{\mathrm{n}}$ of at least a part of (A) or (B).

Consequently, there will always be a step (n) in which $\mathcal{P}^{\mathrm{n}}$, i.e. the set of parts of the set of parts of ... (n times) the set of parts of part of (A) or (B) will have a cardinality higher than (A) or (B), since (A) and (B) are finite and $\mathcal{P}^{\mathrm{n}}$ is infinitely denumerable. Then (A) will be able to construct a surjective map of $\mathcal{P}^{\mathrm{n}}$ on either (A) or (B). This completes the proof.

6.2. About the assessment of probationary spaces

There remain enormous gaps in the present-day knowledge of what universe could really be. For instance, current cosmological theories remain contradictive with astronomical observation (Mitchell, 1995; Bucher and Spergel, 1999; Krauss, 1999 and many other articles). The inconsistency of Lorentz covariance used by Einstein, Minkowski, Mach, Poincaré, Maxwell etc. with Lorentz invariance used by Dirac, Wigner, Feynman, Yang, etc. remains unsolved (Arunasalam, 1997). Quantum mechanics still failing to account for macroscopic phenomena, could be - and has been -interpreted through classical physics (Wesley, 1995), while flaws have been found in so-called "decisive experimental evidence" or classical observations about as fundamental parameters as the Bell inequalities (Wesley, 1994), the velocity of light (Driscoll, 1997), the red shift meaning (Meno, 1998), and others.

More, whether space is independent from matter or matter is deformation of space remains questioned (Krasnoholovets, 1997; Kubel, 1997; Rothwarf, 1998). These discrepancies essentially come from the fact that probationary spaces supporting a number of explicit or implicit assessments have not been clearly identified.

At cosmological scales, the relativity theory places referentials in a undefined space, with undefined gauges nor substrate for the transfer of information and the support of interactions. That matter exists and is spread into this undefined medium is just implicitly admitted without justifications. Here, distances are postulated without reference to objects.

At quantum scales, a probability that objects are present in a certain volume is calculated. But again, nothing is assessed about what are these objects, and what is their embedding medium in which such "volumes" can be found. Furthermore, whether these objects are of a nature similar or different to the nature of their embedding medium has not been addressed. In this case, objects are postulated without reference to distances.

About quantum levels, justifications have been mathematically produced in order to cope with some unexplainable observations, but this does not constitute a proof, per se, since the proof is not independent from the result to be supported. Last, neither the Big Bang energy source nor others have been justified, which led to the theory of an energy of the void: but then, this precludes existence of a true "void". These remarks support the need for finding ultrafilter properties which would be provided to any object and distance from microscopic to cosmic scales in our universe, assuming that it is not composed of separate component with discontinuity nor break of arcwise connectivity.

The next part of this work will provide some logical answers to such problems and derive physical properties of an infering spacetime, with particular reference to the derivation of cosmic scale features from submicroscopic characteristics.

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Appendix

A specimen case of calculation of the dimension of a set whose members are put on the physicist's table like detached parts.

Let $(\mathrm{E}^{\mathrm{e}}_{\mathrm{n}}) = \{\mathrm{aa, ba, ca, bc, ab, cc, ac, bb, cb}\}$. Applying (11.3a) gives several results such as: $\{\mathrm{a, b, bb, ca, cc, ab, ac}\}$ or $\{\mathrm{aa, cc, bb, ca, bc, b, a,}\}$, etc. and $\{\mathrm{aa, bb, cc}\}$. Thus: $\mathrm{diam}(\mathrm{E}^{\mathrm{e}}_{\mathrm{n}}) \subseteq \{\mathrm{aa, bb, cc}\}$ which matches with the diagonal of the Cartesian product $\{\mathrm{a, b, c}\} \times \{\mathrm{a, b, c}\}$. Then, applying (11.3b) gives either $\{\emptyset\}$ or $\{\mathrm{ab}\}$, or $\{\mathrm{bc}\}$, or $\{\mathrm{ac}\}$, that is cobbles having one member of two nuclei as the max representing $\mathrm{diag}(\Pi^{\mathrm{e}}_{\mathrm{n}})$. Since $\mathrm{diag}(\mathrm{E}^{\mathrm{e}}_{\mathrm{n}})$ has three members of two nuclei each, the size ratio is $\varrho = 1/3$ while the number of cobbles tessellating the set is 9. Hence, applying either equations (2.1)–(2.2) or equations (11.2) gives $9 \cdot (1/3)^{\mathrm{e}} = 1$, that is, $\mathrm{e} = \ln 9/\ln 3 = 2$, or $\mathrm{e} = (\ln 9 + 2\ln 2)/\ln 6 = 2$.

The dimension of $(\mathrm{E}^1_3) \times (\mathrm{E}^1_3) = (\mathrm{E}^2_3)$ has been therefore correctly estimated. Had the set been alternatively composed differently, as for instance: $(\mathrm{E}^{\mathrm{e}}_{\mathrm{n}}) = \{\mathrm{aa, ba, ca, bc, ab, cc, ac, bb}\}$ having a lesser number of heterogeneous cobbles, then one would have found: $8 \cdot (1/3)^{\mathrm{e}} = 1$, that is, $\mathrm{e} = 1.89$, a noninteger dimensional exponent, indicating a space with some fractal-like feature.